**Solution exercice 4_1**

(a) Consider that this "hillslope" is flow in a 1D column of soil, and can be treated as a classical Darcy experiment.

H(B) = 2.5 m, H(A) = 1.5 m. The distance between A and B is $(z_B - z_A)/\sin(\alpha)$. So, the total flux per unit width of hillslope is:

$$Q = -KA\frac{dH}{dL} = -KA\frac{\Delta H}{\Delta L} = -K \times 1\ m^2 \times \frac{2.5\ m - 1.5\ m}{1/\sin(\alpha)\ m}$$

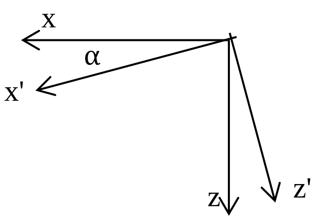
For a sand, we take, for example, $K = 10^{-4}$ m/s, thus $Q = 5.23 \times 10^{-6}\ m^3/s$ per m width of slope.

Observe that, as $\alpha \to 0$, $\sin(\alpha) \to 0$, so $1/\sin(\alpha) \to \infty$. The head difference between A and B is fixed, so the gradient $\to 0$ as $\alpha \to 0$, i.e., no flow. Physically, this means that the point A and B must get further apart as $\alpha \to 0$. In the limit, they are infinitely far apart, so the gradient goes to 0.

Also, H(A) and H(B) are both fixed (i.e., are measured values). As $\alpha$ changes, the pressure head ($p/\rho g$) must also change to keep H constant. This simply means that whatever value of H is measured, since flow is along the slope, the contributions of the forces that drive the flow (gravity force or pressure) are determined by the slope.

(b) **The question examines changing the coordinate system. The mathematics of a 2D coordinate rotation (i.e., changing to downslope coordinates as in this question) are shown below.**

We can transform the original coordinate system into a downslope coordinate system.




Here, the x-z coordinates are rotated through the angle $\alpha$. The rotated coordinates are denoted x'-z'.

For the problem above, in the original x-z coordinates, there is flow in both the x and z directions, whereas in the rotated system there is flow in the x' direction only. The flow in the z' direction is transverse to the downslope layer, and we have assumed that there is no flow in that direction. This simplification will be included below. For the present, we consider the effect of the rotation of the coordinate system on Darcy's law.

There is a subtlety associated with a coordinate rotation[1]. Call the unit vectors in the original coordinate system **i** and **k**, and those in the rotated system **i**' and **k**'. These axes can be obtained from the rotation matrix:

$$\mathbf{R} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

[1] See, e.g., http://mathworld.wolfram.com/RotationMatrix.html

Thus, we have:

$$\mathbf{i}^{'} = \mathbf{Ri} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

and

$$\mathbf{k}^{'} = \mathbf{Rk} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix}.$$

These expressions give the unit vectors in the rotated system in terms of an <u>anti-clockwise rotation of angle α</u>. Note that we can write the relationship between the two sets of unit vectors in a form that will be useful later:

$$\mathbf{i}' = \mathbf{i}\cos\alpha + \mathbf{k}\sin\alpha \qquad \mathbf{i} = \mathbf{i}'\cos\alpha - \mathbf{k}'\sin\alpha$$
$$\mathbf{k}' = -\mathbf{i}\sin\alpha + \mathbf{k}\cos\alpha \qquad \mathbf{k} = \mathbf{i}'\sin\alpha + \mathbf{k}'\cos\alpha$$

Now consider the Darcy flux vector, **q**. <u>Unlike the axes (and unit vectors), **q** is not rotated, but stays fixed in space</u>. Thus, when considering the rotated coordinate system, **q** *appears* to undergo a <u>clockwise rotation of angle α</u>. The rotation matrix for this case is:

$$\mathbf{R} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$$

An arbitrary point (x,z) on this vector has coordinates (x',z') in the rotated coordinate system, i.e.,

$$\begin{bmatrix} x^{'} \\ z^{'} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}.$$

Equally, using this equation we can write:

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}^{-1} \begin{bmatrix} x^{'} \\ z^{'} \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x^{'} \\ z^{'} \end{bmatrix}.$$

Written out, the formulas relating the two coordinate systems are:

$$x' = x\cos\alpha + z\sin\alpha \qquad x = x'\cos\alpha - z'\sin\alpha$$
$$z' = -x\sin\alpha + z\cos\alpha \qquad z = x'\sin\alpha + z'\cos\alpha$$

Darcy's law in 2D is, in the original coordinate system, given as:

$$\mathbf{q} = \mathbf{i}q_x + \mathbf{k}q_z = -K\nabla H = -K(\mathbf{i}\partial H/\partial x + \mathbf{k}\partial H/\partial z).$$

In the new coordinate system, the hydraulic head function, H(x,z), is replaced by H(x'(x,z),z'(x,z)), and so we have:

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial x^{'}}\frac{\partial x^{'}}{\partial x} + \frac{\partial H}{\partial z^{'}}\frac{\partial z^{'}}{\partial x} = \frac{\partial H}{\partial x^{'}}\cos\alpha - \frac{\partial H}{\partial z^{'}}\sin\alpha$$

and

$$\frac{\partial H}{\partial z} = \frac{\partial H}{\partial x^{'}}\frac{\partial x^{'}}{\partial z} + \frac{\partial H}{\partial z^{'}}\frac{\partial z^{'}}{\partial z} = \frac{\partial H}{\partial x^{'}}\sin\alpha + \frac{\partial H}{\partial z^{'}}\cos\alpha$$

Using these results, the Darcy flux, $\mathbf{q} = -K(\mathbf{i}\partial H/\partial x + \mathbf{k}\partial H/\partial z)$, becomes:

$$-\frac{\mathbf{q}}{K} = \left(\mathbf{i}'\cos\alpha - \mathbf{k}'\sin\alpha\right)\left(\frac{\partial H}{\partial x'}\cos\alpha - \frac{\partial H}{\partial z'}\sin\alpha\right) +$$
$$\left(\mathbf{i}'\sin\alpha + \mathbf{k}'\cos\alpha\right)\left(\frac{\partial H}{\partial x'}\sin\alpha + \frac{\partial H}{\partial z'}\cos\alpha\right)$$
$$= \mathbf{i}'\left[\frac{\partial H}{\partial x'}\left(\cos^2\alpha + \sin^2\alpha\right) + \frac{\partial H}{\partial z'}\left(-\cos\alpha\sin\alpha + \cos\alpha\sin\alpha\right)\right] +$$
$$\mathbf{k}'\left[\frac{\partial H}{\partial x'}\left(-\cos\alpha\sin\alpha + \cos\alpha\sin\alpha\right) + \frac{\partial H}{\partial z'}\left(\sin^2\alpha + \cos^2\alpha\right)\right]$$
$$= \mathbf{i}'\frac{\partial H}{\partial x'} + \mathbf{k}'\frac{\partial H}{\partial z'}$$

The above calculation confirms that the <u>form</u> of **q** is <u>invariant under a rotation</u>.

For the case considered in (a), we note that the flux transverse to the impermeable layer is 0, hence $\partial H/\partial z' = 0$. The last line of the above equation becomes:

$$\frac{q}{K} = -i'\frac{\partial H}{\partial x'} = -i'\frac{\Delta H}{\Delta x'} = -i'\frac{1.5 - 2.5}{1/\sin\alpha} = i'\sin\alpha$$

That is, the flow is downslope (in the direction **i**’).

**Solution exercice 4_2**

(a) En plaçant la référence altimétrique au bas de la colonne et en orientant l’axe z positivement vers le haut :

z = 0 cm ➔ h = 0 cm
z = 100 cm ➔ h = 10 cm
H = h + z

Loi de Darcy : $q = -K_S\frac{dH}{dz} = -K_S\frac{\Delta H}{\Delta z}$ (régime permanent)

et donc : $K_S = -q\frac{\Delta z}{\Delta H} = -(-110)\frac{100}{110} = 100$ cm/j = $1.16 \times 10^{-5}$ m/s

Charge de pression en un point situé à 25 cm au-dessus du fond de la colonne : 2.5 cm

(b) l(t) = hauteur d’eau sur la colonne L = hauteur de la colonne

Le flux qui traverse le sol (donné par l’équation de Darcy) doit être égal au flux lié à l’abaissement du niveau d’eau au-dessus du sol :

D’une part (loi de Darcy) : $q = -K_S\frac{\Delta H}{\Delta z} = -K_S\frac{L + l(t)}{L}$

D’autre part : $q = \frac{dl(t)}{dt}$

$$\Rightarrow \frac{dl(t)}{dt} = -K_S\frac{L + l(t)}{L}$$

$$\int_{l_0}^{l}\frac{dl(t)}{L + l(t)} = -\int_{t_0}^{t}\frac{K_S}{L}dt \quad \Rightarrow \quad \left[\ln\left(L + l(t)\right)\right]_{l_0}^{l} = -\frac{K_S}{L}[t]_{t_0}^{t}$$

$$\ln\left(\frac{L + l}{L + l_0}\right) = -\frac{K_S}{L}\cdot(t - t_0) \quad \Rightarrow \quad K_S = -\frac{L}{(t - t_0)}\ln\left(\frac{L + l}{L + l_0}\right)$$

$t - t_o = 3600$ sec  $L = 1$ m  $l_o = 0.1$ m  $l = 0.05$ m

$$K_S = -\frac{1}{3600}\ln\left(\frac{1+0.05}{1+0.1}\right) = 1.29\times10^{-5}\,\mathrm{m/s}$$

**Solution exercice 4_3**

En régime permanent, le flux q dans chacune des couches est identique :

$$q = -K_1\left(\frac{H_1 - H_2}{z_1 - z_2}\right) = -K_2\left(\frac{H_2 - H_3}{z_2 - z_3}\right) \quad \text{ou : } q = -K_1\left(\frac{H_1 - H_2}{L_1}\right) = -K_2\left(\frac{H_2 - H_3}{L_2}\right)$$

Point 1 situé à l'interface eau-couche 1
Point 2 situé à l'interface entre les 2 couches de sol
Point 3 situé au bas de la colonne
$L_1$ : épaisseur couche supérieure (couche 1) ; $L_2$ : épaisseur couche inférieure (couche 2)
$K_1$ et $K_2$ : conductivité hydraulique des couches 1 et 2

Plaçons le plan de référence altimétrique au bas de la colonne et choisissons l'axe des z orienté positivement vers le haut.

$$H_2 = H_1 + \frac{qL_1}{K_1} \quad \text{et :} \quad \frac{qL_2}{K_2} = H_3 - H_2 \qquad \rightarrow \qquad q = \frac{H_3 - H_1}{\frac{L_2}{K_2} + \frac{L_1}{K_1}}$$

a) $q = -3.64 \times 10^{-5}$ cm/s (sens inverse de l'axe des z, soit vers le bas)

b) $H_2 = 181.8$ cm

c) $h_2 = 131.8$ cm

d) Si l'on inverse les couches, le débit ne change pas ; par contre : $H_2 = 18$ cm et $h_2 = -32$ cm.

*On observe que lorsque la couche la plus perméable se trouve au-dessus, l'écoulement est freiné à l'interface, si bien que la charge de pression augmente (elle passe de 100 cm au point 1 à 131.8 cm à l'interface). Le contraire se passe lorsque la couche supérieure est moins conductrice; dans ce cas la pression est dissipée, ici d'une façon telle qu'une pression négative se développe à l'interface. Si cette pression négative excède la valeur de la pression d'entrée d'air (dépression ou succion nécessaire pour que l'air pénètre dans un sol préalablement saturé) de la couche inférieure, cette dernière devient non saturée.*